Variations on character varietes

Joint study group of MPI Leipzig, Leipzig University, and Paderborn University

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In this study group, we first want to get an overview of various different definitions of character varieties, the topologies on them, and the most important subspaces. Then, we contrast the "most flexible" and the "most rigid" cases: discrete and faithful or quasi-Fuchsian representations, where the character varieties are large and rich in structure, vs. examples where it is not clear if the character variety contains more than just a few isolated points. We want to focus on instructive examples, which besides the already mentioned classical quasi-Fuchsian representations could include representations of free groups or Coxeter groups. We also would like to understand compactification techniques and their interpretation as group actions on trees or buildings.

Program

The tentative program is a suggestion and can be freely modified. The speaker can choose the content of their talks based on the topics suggestions, of which there are intentionally too many. We can also spread talks over two sessions if needed - in the program this has already been done for some talks, but we might choose to organize this otherwise. In particular, **the indicated dates of the talks can be changed as required**.

Block I: Basics

Talk 1 (Oct 23, Jacques): Representation and character varieties

The first talk should present the representation variety $\mathrm{Hom}(\Gamma, G)$ from a finitely generated group Γ to a Lie group or and algebraic group G. It should discuss a bit the conjugation action of G on $\mathrm{Hom}(\Gamma, G)$ and explain the main models of the character variety. Explicit examples would greatly help. A possible list of topics could be the following.

- $\mathrm{Hom}(\Gamma, G)$ as a topological space [Mar25, Chapter 2.1]

- $\mathrm{Hom}(\Gamma, G)$ as an analytic/algebraic space [JM87], [LM85], [Sik12]
- The action of G on $\mathrm{Hom}(\Gamma, G)$ [Mar25, Chapter 3]
- The Hausdorff character variety [Mar25, Chapter 4]
- The algebraic character variety [LM85], [Sik12]
- Local obstructions to representation and character varieties [KM98], [KM17]
- Examples [Mar25], [Law07], [Gol22, Chapter 7.3], [HP23], [Sik12], [LM85]

Talk 2 (Oct 30, Benjamin): The case of surface groups

In this talk, we will assume that Γ is the fundamental group of a closed orientable surface S of genus at least 2. In this case, the representation varieties and the character varieties are much more well-behaved and admit extra structures. The motivation to study these character varieties is that they sometimes admit connected components that parametrizes geometric structures on S. A possible list of topics could be the following.

- The smooth structure on the character variety [Mar25, Chapter 4.5]
- The Teichmüller space [BIW14]
- The moduli space of convex real projective structures [Gol90], [CG93]
- Symplectic structure of the character variety [Gol84], [Gol86], [Mar25, Chapter 5]
- Counting of connected components of the character variety [Li93], [Col19, Chapter 1.2]
- Algebraic properties [RBKC96]

Block II: In-depth examples

Talk 3 (Nov 13): Quasi-Fuchsian space and its frontier

The goal of this talk is to present Quasi-Fuchsian space, which is an open set of the character variety of the fundamental group of a closed orientable surface S of genus at least 2 into $\mathrm{SL}(2, \mathbb{C})$. It coincides with the set of convex-cocompact representations from $\pi_1(S)$ to $\mathrm{SL}(2, \mathbb{C})$. Bers uniformization theorem gives a parametrization of Quasi-Fuchsian space by two copies of the Teichmüller space of S. As shown by Sullivan, Quasi-Fuchsian space is the interior of the space of conjugacy classes of discrete and faithful representations from $\pi_1(S)$ to $\mathrm{SL}(2, \mathbb{C})$. The frontier of Quasi-Fuchsian space has also been largely investigated. A possible list of topics could be the following.

- Convex-cocompact representations [Can]
- Sullivan theorem [Sul85]
- Frontier of quasi-fuchsian representations [Can10], [Kap09, Chapter 8]

Talk 4 (Nov 20 – 27): Higher Teichmüller spaces

In this talk, we want to explore higher Teichmüller spaces. These are connected components of the character variety of a surface group where all representations are injective and discrete. The main examples are the Hitchin components and maximal components. Higher Teichmüller spaces are strongly connected with what positive representations. One of the main goal of higher Teichmüller theory is to inteprete them as moduli space of geometric structures on manifold. Finally, higher Teichmüller theory have ramifications outside of surface groups: for hyperbolic n-manifolds for instance. A possible list of topics could be the following.

- Hitchin representations [BIW14], [Can22]
- Maximal representations [BIW14], [Poz]
- Relation to positive representations [FM22], [GLW25], [BGL$^+$24]
- Geometric structures and higher Teichmüller spaces [CG93], [GW12]
- Beyond surface groups [Ben05], [Bar15], [BK25]

Talk 5 (Dec 4): Rigid representations

The goal of this talk is to discuss the problem of rigidity for representations: when can a representation be deformed in representations that are not just conjugate? The main instance of this phenomenon are finite groups are lattices of semisimple Lie groups. For lattices of higher rank Lie groups, even more is true: they are superrigid. On the contrary, surface groups are known to have a very rich theory of deformations. For hyperbolic manifolds of higher dimensions, this lead to questions which are mostly open. A possible list of topics could be the following.

- The case of finite groups
- Local rigidity of lattices [Rag72, Chapters VI and VII]
- Superridity of lattices in higher rank [Mor15, Chapter 16]
- The case of surface groups [KP14]
- Further questions on local rigidity [CLT07], [GM87]

Block III: Compactification

Talk 6 (Dec 11 – Dec 18): Compactification techniques: the classical story

The goal of this talk is to present the compactification of character varieties defined by Morgan-Shalen. Orginally defined for $\mathrm{SL}(2,\mathbb{C})$, this compactification works whenever G has real rank 1. Restricted to the Teichmüller space, this compactification agrees with Thurston's compactification. When G has real rank 1, points in the boundary of this compactification give rise to actions of Γ on real trees. A possible list of topics could be the following.

- The compactification of Morgan-Shalen [MS84], [Ota15]
- Thurston's compactification of Teichmüller space [Ota15], [Bon88]
- The point of view of equivariant Gromov topology [Bes88], [Pau09]
- Actions on real trees [Ota01], [Kap09, Chapters 10 and 11]
- The point of view of harmonic maps [Wol95]
- The topology of this compactification [Wol11]

Talk 7 (Jan 8): Compactification techniques: further ramifications

In this talk, we explore other compactifications and compare them to Morgan-Shalen compactification. The natural extension of the latter to higher rank Lie groups is the Weyl-chamber length compactification, where boundary points give rise to actions of Γ on buildings. The Real Spectrum Compactification is a compactification coming from real algebraic geometry that does not agree with the aforementioned compactifications. A possible list of topics could be the following.

- The Weyl-chamber length compactification [Par12], [BIPP25]
- The Real Spectrum compactification [Bru88], [BIPP21]

Block IV: Further topics

Here are some possible topics for further investigations:

- Anosov representations and their frontiers
- Applications to spectral analysis
- Berkovich analytification of character varieties

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