Axiom A flows and dynamical resonances for Anosov representations

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Abstract This chapter focuses on recent results establishing the existence of dynamically complete geometric structures for Anosov representations, creating a link between higher Teichmüller theory and microlocal analysis. The "geometry at infinity spirit" is materialised in the Axiom A property of analytic contact flows that can be considered as natural higher rank generalisations of the geodesic flows of convex-cocompact locally symmetric spaces of rank one. The three summarised papers treat the projective Anosov, the general Anosov, and the Lorentzian quasi-Fuchsian case, respectively.

Key words: Anosov representations, domains of discontinuity, Axiom A flows, locally homogeneous spaces, exponential mixing, Ruelle zeta function, Ruelle-Pollicott resonances, Lorentzian quasi-Fuchsian manifolds, quantum-classical correspondence

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1 Introduction

The geometry at infinity of a rank one Riemannian locally symmetric space

$$\Sigma = \Gamma\backslash G/K$$

can be described in several equivalent ways. For instance, we call Σ (or Γ) *convex-cocompact* if the discrete torsion-free subgroup $\Gamma \subset G$ of the non-compact semisimple Lie group G preserves and acts cocompactly on a non-empty geodesically convex subset in the Riemannian symmetric space G/K (here $K \subset G$ is a maximal compact subgroup).

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Equivalently, taking on a dynamical systems perspective, Σ (or Γ) is convex-cocompact if, and only if, the geodesic flow

$$\phi^t : T^1\Sigma \to T^1\Sigma$$

on the unit tangent bundle of Σ is an *Axiom A* flow in the sense of Smale [Sma67]. The Axiom A is a combination of a hyperbolicity and an escaping property. To explain it, recall that the *non-wandering set* $\mathrm{NW}(\phi^t)$ of a smooth flow $\phi^t : \mathcal{M} \to \mathcal{M}$ on a smooth manifold $\mathcal{M}$ is the set of all points $p \in \mathcal{M}$ such that for every neighborhood $U \subset \mathcal{M}$ of p and every $T > 0$ there is a $t \geq T$ such that $\phi^t(U) \cap U \neq \emptyset$. Furthermore, recall that a ϕ^t-invariant compact subset $K \subset \mathcal{M}$ is called *hyperbolic* if the restricted tangent bundle $T\mathcal{M}|_K$ admits a $d\phi^t$-invariant splitting

$$T\mathcal{M}|_K = E^{\mathrm{u}} \oplus E^0 \oplus E^{\mathrm{s}} \tag{1}$$

into continuous subbundles such that $E^0(p) = \mathbb{R}X(p)$ for all $p \in K$, where $X := \frac{d}{dt}\big|_{t=0}\phi^t$ is the generating vector field of the flow, and there are $C, c > 0$ such that

$$\left\|d\phi^t(v)\right\| \leq Ce^{-c|t|}\,\|v\| \qquad \forall \begin{cases} t \geq 0,\ v \in E^{\mathrm{s}}, \\ t \leq 0,\ v \in E^{\mathrm{u}}, \end{cases}$$

for some (and hence any) continuous bundle norms on E^{s} and E^{u}, respectively.

Now ϕ^t is called an Axiom A flow if $\mathrm{NW}(\phi^t)$ is compact and hyperbolic and equal to the closure in $\mathcal{M}$ of the union of all periodic orbits. In this case, the dynamics of the flow ϕ^t is sufficiently well-behaved to make it accessible to spectral analysis. Namely, one can assign to ϕ^t an infinite discrete *spectrum* of points in the complex plane called *Ruelle-Pollicott resonances* or *dynamical resonances* by microlocal methods due to Dyatlov-Guillarmou [DG16]. These resonances are defined as the poles of a meromorphic distributional resolvent

$$(X + \lambda)^{-1} : C_c^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M}), \qquad \lambda \in \mathbb{C},$$

of the flow generator X and come with associated finite-dimensional spaces of *resonant states*, which are distributional solutions $u \in \mathcal{D}'(\mathcal{M})$ of the eigenvalue equation

$$(X + \lambda)u = 0.$$

Moreover, in the same context and with the same methods, Dyatlov-Guillarmou [DG18] showed that, thanks to the Axiom A property, the *Ruelle zeta function*

$$\zeta_R(s) := \prod_\gamma \left(1 - e^{-s\ell(\gamma)}\right)^{-1}, \qquad \Re s \gg 0 \tag{2}$$

formed by the lengths $\ell(\gamma)$ of the primitive periodic orbits γ of the flow ϕ^t possesses a meromorphic extension to $\mathbb{C}$. This essentially settled a conjecture of Smale [Sma67] (the orientability assumption in [DG18] was removed by Borns-Weil and Shen [BWS21]).

These results demonstrate that the theory of convex-cocompact rank one locally symmetric spaces has a very feature-rich dynamical and analytical side in addition to the perhaps more well-known geometric and algebraic sides.

If the semisimple Lie group G is of higher real rank, then the notion of an *Anosov representation* $\varrho : \Gamma \to G$ of a word-hyperbolic group Γ (due to Labourie [Lab06] and Guichard-Wienhard [GW12]) generalises the concept of convex-cocompactness.

It is now a natural question if the theory of Anosov representations is equally well linked to spectral theory via Axiom A flows as the theory of convex-cocompact rank one locally symmetric spaces. In the following I illustrate how this question has recently been very positively answered, by summarising the results of three selected research works [DMS25a, DMS25b, DGM25] carried out (partially) in the framework of the Project No. 65 *Resonances for non-compact locally symmetric spaces* within the Priority Programme *Geometry at Infinity* of the German Science Foundation (Deutsche Forschungsgemeinschaft, DFG).

Sections 2, 3 and 4 summarise the results of the papers [DMS25a], [DMS25b], and [DGM25], respectively. The presentation follows the introductions of the respective papers closely, concentrating on a selection of the results. For more details, related and previous works, and background information, I refer the reader to the original papers.

Finally, Section 5 mentions some open questions for future research.

2 Projective Anosov representations

Fix a finite-dimensional real vector space V of dimension $d > 1$. Let $\mathbb{L} \subset \mathbb{P}(V \times V^*)$ denote the open submanifold given by the projectivization of the affine quadric hypersurface $\{(v, \alpha) \in V \times V^* \mid \alpha(v) = 1\}$. The group $\mathrm{SL}(V)$ acts on $\mathbb{P}(V \times V^*)$ by $g \cdot [v : \alpha] = [gv : \alpha \circ g^{-1}]$, preserves $\mathbb{L}$, and commutes with the flow

$$\phi^t : \mathbb{L} \to \mathbb{L}$$

induced by

$$\phi^t(v, \alpha) = (e^t v, e^{-t}\alpha).$$

Moreover, this flow is the Reeb flow of an $\mathrm{SL}(V)$-invariant contact one-form on $\mathbb{L}$ descended from the tautological Liouville one-form on $T^*V = V \times V^*$. The $\mathrm{SL}(V)$-action on $\mathbb{L}$ is transitive and the tangent bundle of $\mathbb{L}$ admits an $\mathrm{SL}(V) \times \{d\phi^t\}_{t\in\mathbb{R}}$-equivariant real analytic splitting

$$T_{[v:\alpha]}\mathbb{L} \simeq \ker(\alpha) \oplus E^0 \oplus \ker(\iota_v) \tag{3}$$

where E^0 is the central flow direction and $\iota_v : V^* \to \mathbb{R}$ is the contraction with $v \in V$.

For $g \in \mathrm{SL}(V)$ we denote by

$$\lambda_1(g) \geq \lambda_2(g) \geq \cdots \geq \lambda_d(g)$$

the logarithms of the moduli of the (complex) eigenvalues of g (with repetitions). Note that any $g \in \mathrm{SL}(V)$ satisfies $\sum_{i=1}^d \lambda_i(g) = 0$, in particular $\lambda_1(g) \geq 0$.

Definition 1 A finitely generated discrete subgroup $\Gamma < \mathrm{SL}(V)$ is called *projective Anosov* if it is Gromov-hyperbolic and there are constants $c, c' > 0$ such that

$$\lambda_1(\gamma) - \lambda_2(\gamma) \geq c|\gamma|_\infty - c' \quad \forall\, \gamma \in \Gamma,$$

where $|\gamma|_\infty = \lim_{n\to+\infty} \frac{|\gamma^n|}{n}$, with $|\cdot|$ the word length with respect to some finite generating set of Γ, the choice of which is irrelevant to the above condition.

Definition 1 is taken from [KP22], in which it is shown to be equivalent to the original definition of a projective Anosov subgroup [Lab06, GW12]. The first main result of [DMS25a] is:

Theorem 1 ([DMS25a, Theorem A]) *Suppose $\Gamma < \mathrm{SL}(V)$ is a torsion-free projective Anosov subgroup.*

1. *There exists a non-empty $\Gamma \times \{\phi^t\}_{t\in\mathbb{R}}$-invariant open subset $\widetilde{\mathcal{M}_\Gamma} \subset \mathbb{L}$ upon which Γ acts freely and properly discontinuously. Therefore, the quotient*

$$\mathcal{M}_\Gamma := \Gamma\backslash\widetilde{\mathcal{M}_\Gamma}$$

is a real analytic locally homogeneous $(\mathrm{SL}(V), \mathbb{L})$-manifold, endowed with a real analytic complete flow $\phi^t : \mathcal{M}_\Gamma \to \mathcal{M}_\Gamma$ preserving a real analytic contact one-form for which ϕ^t is the Reeb flow.

2. *The flow $\phi^t : \mathcal{M}_\Gamma \to \mathcal{M}_\Gamma$ is Axiom A in the sense of Smale and preserves a unique basic hyperbolic set*

$$\mathcal{K}_\Gamma \subset \mathcal{M}_\Gamma.$$

3. *The real analytic splitting (3) descends to a real analytic splitting*

$$T\mathcal{M}_\Gamma = E^{\mathrm{u}} \oplus E^0 \oplus E^{\mathrm{s}}$$

whose restriction to the basic hyperbolic set $\mathcal{K}_\Gamma \subset \mathcal{M}_\Gamma$ is the splitting (1) into unstable-central-stable directions guaranteed by hyperbolicity. In particular, each of the four stable, unstable, central stable, and central unstable foliations along $\mathcal{K}_\Gamma$ are the restriction of global real analytic foliations on $\mathcal{M}_\Gamma$.

In Theorem 1.(1), a locally homogeneous $(\mathrm{SL}(V), \mathbb{L})$-manifold is a real analytic manifold with an $\mathbb{L}$-valued real analytic atlas whose coordinate changes are locally given by the restriction of elements of $\mathrm{SL}(V)$.

Guichard-Wienhard showed [GW12, Prop. 4.3, Rem. 4.12] that every Anosov subgroup $\Gamma < G$ of a semisimple Lie group G may be made projective Anosov after post-composition with a linear representation $G \to \mathrm{SL}(W)$ for some finite-dimensional real vector space W. Therefore, Theorem 1 establishes that all torsion-free Anosov subgroups are monodromy groups of (generally non-compact) Axiom A dynamical systems with analytic regularity. By Selberg's Lemma, the torsion-free property can always be achieved by passing to a finite index subgroup, which preserves the Anosov property [GW12, Cor. 1.3].

A choice of norm on V induces a real analytic trivialization of the principal $\mathbb{R}$-bundle

$$\mathbb{L} \to \mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*)$$

over the space $\mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*)$ of all transverse line/hyperplane pairs $([v],[\alpha]) \in \mathbb{P}(V) \times \mathbb{P}(V^*)$, i.e. those satisfying $\alpha(v) \neq 0$. With respect to this (or any) trivialization, the $\mathrm{SL}(V)$-action becomes a real analytic cocycle

$$\mathscr{H} : \mathrm{SL}(V) \times \mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*) \to \mathbb{R}$$

prescribing the $\mathrm{SL}(V)$-action in the choice of trivialization. This cocycle was initially utilized in the study of discrete group actions by Quint [Qui02a, Qui02b], and after the work [Sam14] appears throughout the literature on Anosov subgroups/homomorphisms.

A projective Anosov subgroup $\Gamma < \mathrm{SL}(V)$ comes with a pair of Γ-equivariant limit maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$, $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$ defined on the Gromov boundary $\partial_\infty\Gamma$. Denoting the complement of the diagonal in $\partial_\infty\Gamma \times \partial_\infty\Gamma$ by $\partial_\infty\Gamma^{(2)}$, Sambarino (see [Sam14, Sam24]) observed that the product map

$$\xi \times \xi^* : \partial_\infty\Gamma^{(2)} \longrightarrow \mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*)$$

can be post-composed by the cocycle $\mathscr{H}$ to obtain a Hölder cocycle

$$\mathscr{H}_\Gamma : \Gamma \times \partial_\infty\Gamma^{(2)} \to \mathbb{R}.$$

Upon leveraging the results of Ledrappier [Led95], Sambarino proved that the induced Γ-action on $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$ is properly discontinuous and cocompact. He denoted the quotient by this action χ_Γ and called the induced flow on χ_Γ the *refraction flow* [Sam24].

The second main result of [DMS25a] establishes that the refraction flow space χ_Γ appears as the basic hyperbolic set of the real analytic contact Axiom A system from Theorem 1.

Theorem 2 ([DMS25a, Theorem B]) *Suppose $\Gamma < \mathrm{SL}(V)$ is a torsion-free projective Anosov subgroup. In the setting of Theorem 1, there is a canonical bi-Hölder continuous homeomorphism*

$$F : \chi_\Gamma \xrightarrow{\simeq} \mathcal{K}_\Gamma$$

such that $F \circ \phi^t_S = \phi^t \circ F$ where $\phi^t_S : \chi_\Gamma \to \chi_\Gamma$ is Sambarino's refraction flow.

As a main application of Theorems 1 and 2, the third main result of [DMS25a] consists of an exponential mixing statement that serves as a *proof of concept* that the aforementioned results open the door to a deeper analysis of the dynamics of the refraction flow associated to projective Anosov subgroups.

Theorem 3 ([DMS25a, Theorem C]) *Suppose $\Gamma < \mathrm{SL}(V)$ is a torsion-free irreducible projective Anosov subgroup. Let $\mathcal{M}_\Gamma$ be the Axiom A dynamical system provided by Theorem 1 with unique basic hyperbolic set $\mathcal{K}_\Gamma \subset \mathcal{M}_\Gamma$.*

For every $0 < \alpha < 1$ and $U \in C^\alpha(\mathcal{K}_\Gamma, \mathbb{R})$, there exist $c_\alpha, C_\alpha > 0$, depending only on U and α, such that for every $F, G \in C^\alpha(\mathcal{K}_\Gamma, \mathbb{R})$ the correlations decay exponentially:

$$\forall\, t \in \mathbb{R} : \left| \int_{z\in\mathcal{K}_\Gamma} F(z)\cdot G(\phi^t(z))\, d\mu_U(z) - \int_{z\in\mathcal{K}_\Gamma} F(z)\, d\mu_U(z) \int_{z\in\mathcal{K}_\Gamma} G(z)\, d\mu_U(z) \right| \\ \leq C_\alpha e^{-c_\alpha |t|} \|F\|_\alpha \|G\|_\alpha .$$

Above, $\|\cdot\|_\alpha$ is the α-Hölder norm and the measure of integration $d\mu_U$ is induced from the unique Gibbs equilibrium state μ_U of maximal topological pressure for the Hölder potential U. In particular, the flow $\phi^t : \mathcal{K}_\Gamma \to \mathcal{K}_\Gamma$ is exponentially mixing for all Gibbs states and Hölder observables, and therefore applying Theorem 2, the refraction flow $\phi^t_S : \chi_\Gamma \to \chi_\Gamma$ is exponentially mixing for all Gibbs states and Hölder observables.

Besides this main application, Theorems 1 and 2 provide a link between the theory of projective Anosov representations and spectral theory, as indicated and motivated in Section 1. The Axiom A system $(\mathcal{M}_\Gamma, \phi^t)$ associated to a non-trivial torsion-free projective Anosov subgroup $\Gamma < \mathrm{SL}(V)$ by Theorem 1 carries a period function

$$\lambda_1 : \Gamma \longrightarrow [0, +\infty) \tag{4}$$

which is conjugacy invariant and positive, i.e. $\lambda_1(\gamma) > 0$ for all non-trivial $e \neq \gamma \in \Gamma$. Here $\lambda_1(\gamma)$ is equal to the natural logarithm of the spectral radius of the unimodular linear map $\gamma \in \mathrm{SL}(V)$. Then the Ruelle zeta function (2) of the Axiom A flow ϕ^t can be written in the form

$$\zeta_R(s) = \prod_{[\gamma]\in[\Gamma]_{\mathrm{prim}}} \left(1 - e^{-s\lambda_1([\gamma])}\right)^{-1}, \tag{5}$$

where $[\Gamma]_{\mathrm{prim}}$ is the set of non-trivial conjugacy classes of primitive elements in the hyperbolic group Γ. The Euler product (5) converges for $\Re(s) > h_{\mathrm{top}}(\Gamma)$, where $h_{\mathrm{top}}(\Gamma)$ denotes the topological entropy of the flow $\phi^t : \mathcal{K}_\Gamma \to \mathcal{K}_\Gamma$ restricted to the unique basic hyperbolic set $\mathcal{K}_\Gamma \subset \mathcal{M}_\Gamma$. Note that $h_{\mathrm{top}}(\Gamma) > 0$ if and only if Γ is non-abelian.

Theorem 4 ([DMS25a, Theorems D, F]) *Suppose $\Gamma < \mathrm{SL}(V)$ is a non-trivial torsion-free projective Anosov subgroup.*

1. *The Ruelle zeta function $\zeta_R(s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = h_{\mathrm{top}}(\Gamma)$.*
2. *If $\Gamma < \mathrm{SL}(V)$ is irreducible, then ζ_R has a zero-free spectral gap: there exists $\varepsilon > 0$ such that ζ_R is holomorphic and nowhere vanishing in the strip*

$$h_{\mathrm{top}}(\Gamma) - \varepsilon < \mathrm{Re}(s) < h_{\mathrm{top}}(\Gamma).$$

3. *For each smooth potential $U \in C^\infty_c(\mathcal{M}_\Gamma, \mathbb{C})$ supported in a sufficiently tight open neighborhood around the basic hyperbolic set $\mathcal{K}_\Gamma$, there is an infinite discrete spectrum $\mathcal{R}^{\mathbf{X}}_\Gamma \subset \mathbb{C}$ of* Ruelle-Pollicott resonances *associated to the operator*

$$\mathbf{X} := -X + U,$$

where $X = \frac{d}{dt}\big|_{t=0}\phi^t$ is the generator of the flow ϕ^t.

4. *For every resonance* $\lambda_0 \in \mathcal{R}_\Gamma^{\mathbf{X}}$*, there exist pairs of non-zero distributions u (respectively v) called* resonant states *(respectively* coresonant states*) defined in a relatively compact open neighborhood of* $\mathcal{K}_\Gamma$ *and satisfying*

$$\mathbf{X}u = \lambda_0 u, \qquad \mathbf{X}^* v = \overline{\lambda}_0 v,$$

where $\mathbf{X}^* := X + \overline{U}$ *is the formal adjoint of* $\mathbf{X}$*. Their product* $u \cdot \overline{v}$ *is a well-defined* ϕ^t*-invariant distribution satisfying* $\operatorname{supp}(u \cdot \overline{v}) \subset \mathcal{K}_\Gamma$.

2.1 Applications to Finsler and pseudo-Riemannian geodesic flows

In [DMS25a, Section 6], we apply our Axiom A framework in two specific settings: $\mathbb{H}^{p,q}$-convex cocompact subgroups $\Gamma < \mathrm{SO}(p, q+1)$ developed by Danciger-Guéritaud-Kassel [DGK18], and strictly convex divisible domains developed by Benoist [Ben08]. In particular, the space-like geodesic flow for convex cocompact $\mathbb{H}^{p,q}$-manifolds and the Benoist-Hilbert geodesic flow for strictly convex $\mathbb{RP}^d$-manifolds are shown to be exponentially mixing.

Suppose $\Gamma < \mathrm{SL}(V)$ divides a strictly convex divisible domain $C_\Gamma \subset \mathbb{P}(V)$ and

$$\mathcal{N}_\Gamma = \Gamma \backslash C_\Gamma$$

is the associated closed strictly convex real projective manifold. The Hilbert metric induces a $C^{1,\alpha}$-norm on $T\mathcal{N}_\Gamma \setminus \mathcal{Z}$ where $\mathcal{Z} \subset T\mathcal{N}_\Gamma$ denotes the zero section. Let

$$\mathbb{S}\mathcal{N}_\Gamma := (T\mathcal{N}_\Gamma \setminus \mathcal{Z})/\mathbb{R}_+^\times$$

denote the conformal sphere bundle equipped with the Benoist-Hilbert geodesic flow

$$\phi_{BH}^t : \mathbb{S}\mathcal{N}_\Gamma \to \mathbb{S}\mathcal{N}_\Gamma.$$

As proved by Benoist [Ben04], the Benoist-Hilbert geodesic flow is a $C^{1,\alpha}$ topologically transitive Anosov flow which is C^2 if and only if there is an isometry $C_\Gamma \simeq \mathbb{H}^d$. It turned out to be exponentially mixing as a corollary of Theorem 3:

Theorem 5 ([DMS25a, Theorem G]) *The Benoist-Hilbert geodesic flow*

$$\phi_{BH}^t : \mathbb{S}\mathcal{N}_\Gamma \to \mathbb{S}\mathcal{N}_\Gamma$$

mixes exponentially for all Hölder observables with respect to every Gibbs equilibrium state with Hölder potential.

The Benoist-Hilbert geodesic flow had long been out of reach of quantitative mixing results because of its low regularity (c.f. Liverani [Liv04] establishing exponential mixing for C^4 Anosov flows). Our proof trades the real analytic manifold $\mathbb{S}\mathcal{N}_\Gamma$ and the $C^{1,\alpha}$ flow ϕ_{BH}^t for a Lipschitz submanifold of a real analytic contact Axiom A system whose stable and unstable foliations are the restriction of global real analytic foliations.

In a somewhat different direction, concerning the pseudo-Riemannian symmetric space $\mathbb{H}^{p,q}$, Danciger-Guéritaud-Kassel [DGK18] defined convex-cocompact subgroups $\Gamma < \mathrm{SO}(p, q+1)$ and constructed corresponding non-empty domains of proper discontinuity $\Omega_\Gamma^{\mathrm{DGK}} \subset \mathbb{H}^{p,q}$. The space-like geodesic flow

$$\phi^t : T^1\left(\Gamma\backslash\Omega_\Gamma^{\mathrm{DGK}}\right) \to T^1\left(\Gamma\backslash\Omega_\Gamma^{\mathrm{DGK}}\right)$$

is incomplete in general, while the Axiom A system $\mathcal{M}_\Gamma$ we construct in Theorem 1 satisfies $T^1(\Gamma\backslash\Omega_\Gamma^{\mathrm{DGK}}) \subset \mathcal{M}_\Gamma$ and contains the saturation of $T^1(\Gamma\backslash\Omega_\Gamma^{\mathrm{DGK}})$ with respect to the foliation of $\mathcal{M}_\Gamma$ by flow lines. As the final sample result of [DMS25a], I mention

Theorem 6 ([DMS25a, Theorem H]) *Suppose that Γ is irreducible. Then the restriction of the (possibly incomplete) space-like geodesic flow*

$$\phi^t : T^1\left(\Gamma\backslash\Omega_\Gamma^{\mathrm{DGK}}\right) \to T^1\left(\Gamma\backslash\Omega_\Gamma^{\mathrm{DGK}}\right)$$

to the basic hyperbolic set $\mathcal{K}_\Gamma \subset T^1\left(\Gamma\backslash\Omega_\Gamma^{\mathrm{DGK}}\right)$ mixes exponentially for all Hölder observables with respect to every Gibbs equilibrium state with Hölder potential.

3 General Anosov representations

While the paper [DMS25a] considers only projective Anosov representations for the sake of clarity (one does not need to introduce any Lie theory in this case, which is a huge technical simplification), we treat the general case in [DMS25b]. Here I recall the results of the latter work in a slightly compressed form.

As mentioned in Section 1, the concept of a Θ-Anosov subgroup $\Gamma < G$ of a semisimple real Lie group G (i.e., the image of a Θ-Anosov representation with respect to a subset $\Theta \subset \Delta$ of a simple system) was introduced by Labourie [Lab06] and generalized by Guichard-Wienhard [GW12]. As the name suggests, Labourie's original approach was intimately related to uniformly hyperbolic smooth dynamical systems (a.k.a. Anosov flows), but modern definitions [KLP17, GGKW17, BPS19, KP22] no longer appeal to flows, instead casting the Θ-Anosov property in terms of growth rates of singular or eigenvalues.

If G has real rank one, a subgroup $\Gamma < G$ is Θ-Anosov if and only if it is convex cocompact [GW12, Thm. 5.15], and if Γ is torsion-free the rank one Riemannian locally symmetric space is a geometric structure for Γ. The unit tangent bundle of the latter carries the geodesic flow, featuring a similar uniformly hyperbolic behavior as an Anosov flow (Smale's Axiom A, see Section 1).

In higher rank, the construction of geometric structures associated to (torsion-free) Θ-Anosov subgroups $\Gamma < G$ is a key challenge in the field [Kas18, Wie18] with many results (e.g. [GW12, KLP18, GGKW17, CS24, NR24]), yet a systematic construction which always produces non-empty locally homogeneous manifolds had remained elusive. In [DMS25b], we provide such a construction. The obtained manifolds carry analytic Axiom A flows or partially hyperbolic multiflows, respectively, reaffirming the original link between Anosov subgroups and smooth hyperbolic dynamics.

The analog of the space $(\mathcal{M}, \phi^t)$ from Theorem 1 when $\Gamma < G$ is a torsion-free Θ-Anosov subgroup is a family of Axiom A systems $(\mathcal{M}_{\mathbf{b}}, \phi^t_{\mathbf{b}})$ indexed by additive characters $\mathbf{b} : L_\Theta \to \mathbb{R}$ in an open convex cone in $\mathrm{Hom}(L_\Theta, \mathbb{R}) \simeq \mathbb{R}^{|\Theta|}$, where $L_\Theta < G$ is the standard Levi subgroup associated to $\Theta \subset \Delta$.

Sambarino [Sam24] used additive characters on L_Θ to construct families of refraction flows associated to any Θ-Anosov subgroup, and we will show that every refraction flow is bi-Hölder conjugate to the restriction of the Axiom A flow $\phi^t_{\mathbf{b}}$ to its unique basic hyperbolic set $\mathcal{K}_{\mathbf{b}} \subset \mathcal{M}_{\mathbf{b}}$, in particular arriving at the same open convex cone of admissible parameters $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R})$ as Sambarino.

Our starting point is the observation that the differential $d_e\mathbf{b}$ canonically extends to a linear form $\beta : \mathfrak{g} \to \mathbb{R}$, and the coadjoint G-orbit of β is generically isomorphic to G/L_Θ. Simultaneously, $\mathbf{b} : L_\Theta \to \mathbb{R}$ defines a homogeneous $\mathbb{R}$-affine line bundle $\mathbb{L}_{\mathbf{b}}$ over this coadjoint orbit, and it is in the total space of $\mathbb{L}_{\mathbf{b}}$ where we shall construct a non-empty domain of proper discontinuity for the Γ-action. We now turn to the details of this construction and the statements of our main results.

Throughout, G denotes a non-compact real semisimple Lie group with Lie algebra $\mathfrak{g}$ and $\Theta \subset \Delta$ a non-empty subset of simple restricted roots defining associated opposite proper parabolic subgroups $P^+_\Theta, P^-_\Theta < G$ with standard Levi intersection $L_\Theta = P^+_\Theta \cap P^-_\Theta$.

3.1 Setup

The construction of the homogeneous spaces studied in [DMS25b] can be thought of as reverse-engineering the unit tangent bundle $T^1\mathbb{X}$ of a rank-one symmetric space $\mathbb{X}$ from the projections $T^1\mathbb{X} \to \partial_\infty\mathbb{X}$ sending a unit tangent vector $v \in T^1_x\mathbb{X}$ to the future and past endpoints $v^\pm \in \partial_\infty\mathbb{X}$ of the geodesic initiated by v. We replace the visual boundary $\partial_\infty\mathbb{X}$ with a pair of *opposite flag manifolds* $\mathcal{F}^\pm \simeq G/P^\pm_\Theta$. The image in $\partial_\infty\mathbb{X} \times \partial_\infty\mathbb{X}$ of the projection $v \mapsto (v^+, v^-)$ will be replaced with the *transverse flag space* $\mathcal{F}^+ \overset{\pitchfork}{\times} \mathcal{F}^- \subset \mathcal{F}^+ \times \mathcal{F}^-$, the unique open G-orbit in $\mathcal{F}^+ \times \mathcal{F}^- \simeq G/P^+_\Theta \times G/P^+_\Theta$ which can be equivalently described as the homogeneous space $\mathcal{F}^+ \overset{\pitchfork}{\times} \mathcal{F}^- \simeq G/L_\Theta$.

3.1.1 Hyperbolic homogeneous flows

By a homogeneous flow on a homogeneous space $X = G/H$ we mean a flow $\phi^t : X \to X$ defined for all $t \in \mathbb{R}$ that commutes with the G-action. We will study the case where the orbit space of ϕ^t is the homogeneous space G/L_Θ, leading us to consider subgroups $H_{\mathbf{b}} := \ker \mathbf{b}$ for some non-zero additive character $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R})$. From the structure of Levi subgroups, we find a canonical choice of an element $Z_{\mathbf{b}} \in \mathfrak{z}(\mathfrak{l}_\Theta)$ in the center of the Lie algebra of L_Θ such that $d_e\mathbf{b}(Z_{\mathbf{b}}) = 1$. This allows us to equip the homogeneous space $\mathbb{L}_{\mathbf{b}} := G/H_{\mathbf{b}}$ with the flow

$$\phi_{\mathbf{b}}^t : \begin{cases} \mathbb{L}_{\mathbf{b}} & \to & \mathbb{L}_{\mathbf{b}} \\ gH_{\mathbf{b}} & \mapsto & g\exp(tZ_{\mathbf{b}})H_{\mathbf{b}}, \end{cases}$$

which is a right action of the one-parameter subgroup $A_{\mathbf{b}} := \exp(\mathbb{R}Z_{\mathbf{b}}) < Z(L_\Theta)$.

From the embedding of G/L_Θ into $G/P_\Theta^+ \times G/P_\Theta^-$, we obtain a decomposition

$$\mathrm{T}\mathbb{L}_{\mathbf{b}} = E^- \oplus E^0 \oplus E^+, \tag{6}$$

where the rank one bundle E^0 is tangent to the flow and $E^\pm$ is sent isomorphically onto $T(G/P_\Theta^\pm)$ by the differential of the projection $\mathbb{L}_{\mathbf{b}} = G/H_{\mathbf{b}} \to G/L_\Theta$.

Definition 2 The pair $(\mathbb{L}_{\mathbf{b}}, \phi_{\mathbf{b}}^t)$ will be referred to as a *hyperbolic homogeneous flow*.

3.1.2 Multiflows

Hyperbolic homogeneous flows with a given orbit space G/L_Θ form a family indexed by $\mathrm{Hom}(L_\Theta, \mathbb{R})$. The Levi subgroup L_Θ splits as a direct product $L = A_\Theta M_\Theta$, where $A_\Theta < Z(L_\Theta)$ is isomorphic to the abelian Lie group $\mathbb{R}^{|\Theta|}$, M_Θ is a reductive subgroup, and the restriction map $\mathrm{Hom}(L_\Theta, \mathbb{R}) \to \mathrm{Hom}(A_\Theta, \mathbb{R})$ is an isomorphism. Equivalently, these subgroups can be described as factors in the Langlands decompositions $P_\Theta^\pm = M_\Theta A_\Theta N_\Theta^\pm$. The homogeneous space $\mathbb{W}_\Theta := G/M_\Theta$ comes with an action of $A_\Theta \simeq \mathbb{R}^{|\Theta|}$ on the right given by $(gM_\Theta) \cdot h := ghM_\Theta$ for $h \in A_\Theta$. For $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R}) \setminus \{0\}$, the normal subgroup $M_\Theta \lhd H_{\mathbf{b}}$ satisfies $H_{\mathbf{b}}/M_\Theta \simeq A_\Theta/A_{\mathbf{b}} \simeq \mathbb{R}^{|\Theta|-1}$, and defining the projection $p_{\mathbf{b}} : \mathbb{W}_\Theta = G/M_\Theta \to G/H_{\mathbf{b}} = \mathbb{L}_{\mathbf{b}}$, we find that

$$p_{\mathbf{b}}(x \cdot h) = \phi_{\mathbf{b}}^{\mathbf{b}(h)}\big(p_{\mathbf{b}}(x)\big) \qquad \forall\, x \in \mathbb{W}_\Theta, h \in A_\Theta.$$

Definition 3 The right A_Θ-action on $\mathbb{W}_\Theta = G/M_\Theta$ will be referred to as a *multiflow*.

Dynamically, the fact that $p_{\mathbf{b}} : \mathbb{W}_\Theta \to \mathbb{L}_{\mathbf{b}}$ is a principal $H_{\mathbf{b}}/M_\Theta \simeq A_\Theta/A_{\mathbf{b}} \simeq \mathbb{R}^{|\Theta|-1}$-bundle means that $\mathbb{L}_{\mathbf{b}}$ is obtained from $\mathbb{W}_\Theta$ by “quotienting out all but one flow direction.”

For a Θ-Anosov subgroup $\Gamma < G$, we will see that there exists a non-empty domain of proper discontinuity $\widetilde{\mathcal{N}} \subset \mathbb{W}_\Theta$ such that the $\mathbb{R}^{|\Theta|-1} \simeq A_\Theta/A_{\mathbf{b}}$-action is proper on the Γ-quotient $\Gamma\backslash\widetilde{\mathcal{N}}$ and the quotient $\Gamma\backslash\widetilde{\mathcal{N}}/\mathbb{R}^{|\Theta|-1}$ is an Axiom A system.

3.1.3 (G, X)-structures for Anosov subgroups

Given a G-homogeneous space $X = G/H$ and a manifold $\mathcal{M}$, a (G, X)-structure on $\mathcal{M}$ is a maximal atlas of X-valued charts on $\mathcal{M}$ such that the transition maps are G-valued and locally constant. Starting with a discrete subgroup $\Gamma < G$, one way of producing a (G, X)-manifold is to find a non-empty open Γ-invariant subset $U \subset X$ on which the Γ-action is free and properly discontinuous. The quotient manifold

$$\mathcal{M}_\Gamma := \Gamma\backslash U$$

carries a (G, X)-structure which we call *uniformized* (this is referred to as “Type U” in [Kas18]).

Any (G, X)-manifold inherits all local structures on X preserved by G. In the case of $X = \mathbb{L}_{\mathbf{b}}$ for some $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R}) \setminus \{0\}$, this means that a $(G, \mathbb{L}_{\mathbf{b}})$-manifold comes with a local flow, i.e. a vector field. Similarly, a $(G, \mathbb{W}_\Theta)$-structure defines a local A_Θ-action, i.e. a family of $|\Theta|$ commuting vector fields.

Definition 4 A (G, X)-structure with $X = \mathbb{L}_{\mathbf{b}}$ or $X = \mathbb{W}_\Theta$ is called *dynamically complete* if the induced vector fields are complete.

In the uniformized case, this means that the open subset $U \subset \mathbb{L}_{\mathbf{b}}$ (resp. $U \subset \mathbb{W}_\Theta$) is $\phi^t_{\mathbf{b}}$-invariant (resp. A_Θ-invariant). The history of (G, X)-structures associated to Anosov subgroups is rich and well documented, see [Kas18, Wie18, CS24] and references therein. Before [DMS25a], dynamical (G, X)-structures for higher rank Anosov subgroups were known to exist (e.g. [GW08]) but were rather rare among known examples. The main result of [DMS25b] is that dynamical (G, X)-structures with strong hyperbolicity properties exist in abundance.

3.2 Results

Consider a Θ-Anosov subgroup $\Gamma < G$. The dynamics of the Γ-action on the flow space $\mathbb{L}_{\mathbf{b}}$ will depend on the character $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R})$, and especially on its relation to the Θ-*limit cone* $\mathcal{L}_\Theta(\Gamma) \subset \mathfrak{a}^+_\Theta$ (the projection of the *Benoist limit cone* $\mathcal{L}(\Gamma)$ to the Lie algebra $\mathfrak{a}_\Theta$ of A_Θ). Consider the dual cone

$$\mathcal{L}_\Theta(\Gamma)^* = \left\{\beta \in \mathfrak{a}^*_\Theta \,\middle|\, \beta|_{\mathcal{L}_\Theta(\Gamma)} \geq 0\right\},$$

whose interior $\mathrm{Int}(\mathcal{L}_\Theta(\Gamma)^*)$ is non-empty because $\mathcal{L}_\Theta(\Gamma)$ is acute, and identify the differential $d_e\mathbf{b} \in \mathfrak{l}^*_\Theta$ with its restriction to $\mathfrak{a}_\Theta \subset \mathfrak{l}_\Theta$, where $\mathfrak{l}_\Theta$ is the Lie algebra of L_Θ.

3.2.1 Uniformized (G, X)-structures

The first result of [DMS25b] establishes the existence of (G, X)-structures and hyperbolic homogeneous flows, generalizing Theorem 1.1.

Theorem 7 ([DMS25b, Theorem A]) *Let $\Gamma < G$ be Θ-Anosov, and let $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R})$. If $d_e\mathbf{b} \in \mathrm{Int}(\mathcal{L}_\Theta(\Gamma)^*)$, then there is a non-empty $\Gamma \times \phi^t_{\mathbf{b}}$-invariant open set $\widetilde{\mathcal{M}_{\mathbf{b}}} \subset \mathbb{L}_{\mathbf{b}}$ on which Γ acts properly discontinuously, yielding a dynamically complete uniformized $(G, \mathbb{L}_{\mathbf{b}})$-manifold*

$$\mathcal{M}_{\mathbf{b}} := \Gamma\backslash\widetilde{\mathcal{M}_{\mathbf{b}}}$$

when Γ is torsion-free.

It is to be noted that the action of Γ on $\mathbb{L}_{\mathbf{b}}$ is almost never proper, the only notable exception being the case of a group G of real rank one (then $\mathbb{L}_{\mathbf{b}} \simeq T^1\mathbb{X}$). The fact that the multiflow space $\mathbb{W}_\Theta$ fibers over $\mathbb{L}_{\mathbf{b}}$ immediately yields the following consequence:

Theorem 8 ([DMS25b, Corollary B]) *Let $\Gamma < G$ be Θ-Anosov. There is a non-empty $\Gamma \times A_\Theta$-invariant open set $\widetilde{\mathcal{N}} \subset \mathbb{W}_\Theta$ on which Γ acts properly discontinuously, yielding a dynamically complete uniformized $(G, \mathbb{W}_\Theta)$-manifold*

$$\mathcal{N} := \Gamma \backslash \widetilde{\mathcal{N}}$$

when Γ is torsion-free.

For each $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R})$ with $d_e\mathbf{b} \in \mathrm{Int}(\mathcal{L}_\Theta(\Gamma)^*)$, the projection

$$\mathcal{N} \to \mathcal{M}_{\mathbf{b}}$$

induced by the canonical projection $\mathbb{W}_\Theta \to \mathbb{L}_{\mathbf{b}}$ is a right principal $\mathbb{R}^{|\Theta|-1} \simeq A_\Theta/A_{\mathbf{b}}$-bundle.

The quotient manifolds $\mathcal{M}_{\mathbf{b}}$ are usually non-compact, but we will see that in each case the closure of the set of periodic orbits is compact and equal to the non-wandering set of the flow. A helpful example is that of quasi-Fuchsian surface groups $\Gamma < \mathrm{PSL}(2, \mathbb{C})$, where the locally homogeneous manifolds are non-compact but have compact cores.

In real rank one, or more generally when $|\Theta| = 1$, the spaces $\mathbb{L}_{\mathbf{b}}$ and $\mathbb{W}_\Theta$ are identical. In higher rank, the situation when Θ is the entire set Δ of simple restricted roots stands out because of the properness of the G-action on $\mathbb{W}_\Delta$.

The homogeneous space $\mathbb{W}_\Theta = G/M_\Theta$ often appears in the study of Anosov subgroups in the form of a Hopf parametrization $\mathbb{W}_\Theta \simeq G/L_\Theta \times \mathfrak{a}_\Theta$ with a cocycle defining the G-action on the right hand side, similarly for $\mathbb{L}_{\mathbf{b}} \simeq G/L_\Theta \times \mathbb{R}$. The novelty in Theorem 7 and Corollary 8 is the discovery of the domains of proper discontinuity $\widetilde{\mathcal{M}}_{\mathbf{b}} \subset \mathbb{L}_{\mathbf{b}}$, $\widetilde{\mathcal{N}} \subset \mathbb{W}_\Theta$.

3.2.2 Dynamics of quotient flows

Next, we turn to the dynamics of the flows defined by $(G, \mathbb{L}_{\mathbf{b}})$-structures obtained in Theorem 7, establishing Smale's Axiom A.

Theorem 9 ([DMS25b, Theorem C]) *Let $\Gamma < G$ be a torsion-free Θ-Anosov subgroup, and let $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R})$ be such that $d_e\mathbf{b} \in \mathrm{Int}(\mathcal{L}_\Theta(\Gamma)^*)$. Consider the $(G, \mathbb{L}_{\mathbf{b}})$-manifold $\mathcal{M}_{\mathbf{b}}$ with its flow $\phi^t_{\mathbf{b}}$ obtained from Theorem 7.*

1. *The real analytic flow $\phi^t_{\mathbf{b}} : \mathcal{M}_{\mathbf{b}} \to \mathcal{M}_{\mathbf{b}}$ is an Axiom A flow.*
2. *The non-wandering set of $\phi^t_{\mathbf{b}}$ consists of a unique basic set $\mathcal{K}_{\mathbf{b}} \subset \mathcal{M}_{\mathbf{b}}$, and the splitting of $T\mathcal{M}_{\mathbf{b}}|_{\mathcal{K}_{\mathbf{b}}}$ into stable/neutral/unstable subbundles provided by the Axiom A property is given by first descending the G-invariant splitting (6) to the global real analytic $\phi^t_{\mathbf{b}}$-invariant splitting $T\mathcal{M}_{\mathbf{b}} = E^-_{\Gamma,\mathbf{b}} \oplus E^0_{\Gamma,\mathbf{b}} \oplus E^+_{\Gamma,\mathbf{b}}$, and then restricting to the basic hyperbolic set $\mathcal{K}_{\mathbf{b}}$.*

3. *The real analytic $\phi^t_{\mathbf{b}}$-invariant* 1*-form $\tau_{\mathbf{b}}$ which vanishes on $E^-_{\Gamma,\mathbf{b}} \oplus E^+_{\Gamma,\mathbf{b}}$ and is equal to* 1 *on the generator of $\phi^t_{\mathbf{b}}$ is a contact* 1*-form if and only if $\mathrm{B}_{\mathfrak{g}}(d_e\mathbf{b}, w_\alpha) \neq 0$ for every fundamental weight w_α with $\alpha \in \Theta$, where $\mathrm{B}_{\mathfrak{g}}$ is the Killing form.*

For each character $\mathbf{b} \in \mathrm{Hom}(L_\Theta, \mathbb{R})$ as in Theorem 9, Sambarino [Sam24] defined a model of the Gromov geodesic flow of Γ given by a generalised *refraction flow* $\psi^t_{\mathbf{b}}$. As the last sample result of [DMS25b], I mention:

Theorem 10 ([DMS25b, Theorem D]) *In the situation of Theorem 9, the restriction of the Axiom A flow $\phi^t_{\mathbf{b}} : \mathcal{M}_{\mathbf{b}} \to \mathcal{M}_{\mathbf{b}}$ to its basic set $\mathcal{K}_{\mathbf{b}} \subset \mathcal{M}_{\mathbf{b}}$ is bi-Hölder-conjugate to the refraction flow $\psi^t_{\mathbf{b}}$.*

Theorems 9 and 10 are direct generalizations of Theorems 1 and 2.

4 Lorentzian quasi-Fuchsian representations

While the works [DMS25a, DMS25b] were motivated by the general goal of introducing dynamically complete geometric structures and dynamical resonances for Anosov representations, the work [DGM25] on Lorentzian quasi-Fuchsian representations in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ was motivated by the goal of understanding the spectral theory of the Lorentzian quasi-Fuchsian locally anti-de Sitter manifolds associated to these representations. From a mathematical physics perspective, this spectral theory has a classical as well as quantum side. The classical side belongs to the general framework studied in [DMS25a, DMS25b], here the considered Axiom A flow is (a complete extension of) the *space-like geodesic flow* on the Lorentzian unit tangent bundle, and the spectral theory of this flow can be understood with essentially the same methods as that of the Axiom A flows appearing in [DMS25a, DMS25b]. The quantum side is the spectral theory of the pseudo-Riemannian Laplace-Beltrami operator Δ (often written $\square$ and called the d'Alembertian), which in this case turned out to be widely unknown, in stark contrast to the Riemannian case.

Let us now turn to the summary of the results of the work [DGM25].

4.1 Spectra of Lorentzian quasi-Fuchsian manifolds

Three-dimensional Lorentzian quasi-Fuchsian manifolds are analogues of the quasi-Fuchsian Riemannian hyperbolic 3-manifolds obtained by quasiconformal deformation of a co-compact Fuchsian subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ inside $\mathrm{SL}(2,\mathbb{C})$, which are complete infinite volume Riemannian manifolds diffeomorphic to a cylinder $(\Gamma\backslash\mathbb{H}^2) \times (-1,1)$. The Lorentzian version is a globally hyperbolic Lorentzian manifold that is also diffeomorphic to such a cylinder, but is not complete: it is obtained by quotienting an open subset Ω_Γ of the anti-de Sitter space AdS_3 on which Γ acts properly discontinuously.

In a nutshell, it is shown in [DGM25] that on a quasi-Fuchsian Lorentzian manifold $M = \Gamma\backslash\Omega_\Gamma$ equipped with its natural Lorentzian metric g induced from AdS_3:

(I) The incomplete geodesic flow on the space-like unit tangent bundle T^1M extends to a complete flow on a smooth manifold $\mathcal{M} \supset T^1M$ and, using [DG16], it follows that the resolvent

$$R_X(\lambda) = (X+\lambda)^{-1} = \int_0^\infty e^{-t(X+\lambda)}dt : C_c^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$$

of the flow generator X continues meromorphically from $\mathrm{Re}(\lambda) > 0$ to $\lambda \in \mathbb{C}$, with poles forming a discrete spectrum of *Ruelle resonances*. The elements in the range of the residue at a resonance $\lambda_0 \in \mathbb{C}$ form a finite-dimensional space and those in $\ker(X+\lambda_0)$ are called the *Ruelle resonant states*.

(II) The Poincaré series

$$\mathcal{D}_\lambda(x,x') = \sum_{\gamma \in \Gamma^c_{x,x'}} e^{-\lambda d(x,\gamma x')}$$

for $x, x' \in \Omega_\Gamma$ is holomorphic in $\mathrm{Re}(\lambda) > 1$ and extends meromorphically to $\mathbb{C}$. Here $\Gamma^c_{x,x'}$ is the set of $\gamma \in \Gamma$ such that x and $\gamma x'$ are joined by a (necessarily unique) spacelike geodesic and the distance $d(x,\gamma x')$ is the length of this geodesic.

(III) The pseudo-Riemannian Laplacian $\Box_g$ on M admits an analytic resolvent in the region $\{\mathrm{Re}(\lambda) > -1\}$

$$R_{\Box_g}(\lambda) : C_c^\infty(M) \to \mathcal{D}'(M)$$

inverting $(\Box_g + \lambda(\lambda+2))$, which continues meromorphically to $\lambda \in \mathbb{C}$. Here the quadratic parametrization $\lambda(\lambda+2)$ (rather than just λ) of the spectral parameter is compatible with the standard convention in the Riemannian case and makes it easier to directly compare the spectral parameter of the second order operator $\Box_g$ with that of the first order operator X. The poles of $R_{\Box_g}(\lambda)$ form a discrete spectrum of *quantum resonances*. The elements in the range of the residue at a resonance $\lambda_0 \in \mathbb{C}$ form a finite-dimensional space and those in $\ker(\Box_g + \lambda_0(\lambda_0+2))$ are called the *quantum resonant states*. Moreover, we prove a version of quantum-classical correspondence between the two resolvents of $\Box_g$ and of X and analyse the first poles.

Let us now briefly summarise some selected details and noteworthy remarks.

4.2 Extension of the space-like geodesic flow and quantum-classical correspondence

The three-dimensional anti-de Sitter space

$$\mathrm{AdS}_3 = \{x \in \mathbb{R}^4 \mid q(x) = -1\}, \qquad q(x) := -x_1^2 - x_2^2 + x_3^2 + x_4^2$$

equipped with the Lorentzian metric given by $q|_{\mathrm{AdS}_3}$ is isometric to the Lie group $G = \mathrm{SL}(2,\mathbb{R})$ equipped with its Killing Lorentzian metric (up to a normalisation constant). The group $G \times G$ acts on G as isometries by

$$(h_1, h_2) \cdot h = h_1 h h_2^{-1}$$

with isotropy group the diagonal subgroup $G_d := \{(h, h) \in G \times G\}$. A quasi-Fuchsian group Γ has a non-empty Γ-invariant proper open subset $\Omega_\Gamma \subset \mathrm{AdS}_3$ on which Γ acts properly discontinuously and freely, defining the quasi-Fuchsian manifold $M := \Gamma\backslash\Omega_\Gamma$. These *quasi-Fuchsian* groups, studied initially in [Mes07], are precisely the groups $\Gamma \subset G \times G$ of the form

$$\Gamma = \{(\rho_1(\gamma), \rho_2(\gamma)) \mid \gamma \in \pi_1(\Sigma)\},$$

where $\rho_1, \rho_2 : \pi_1(\Sigma) \to G = \mathrm{SL}(2, \mathbb{R})$ are the holonomy representations of the fundamental group of a closed orientable surface Σ of genus ≥ 2 equipped with two hyperbolic metrics.

The *space-like unit tangent bundle* $T^1M \subset TM$ of a quasi-Fuchsian manifold $M = \Gamma\backslash\Omega_\Gamma$ is formed by all tangent vectors of length 1 with respect to the Lorentzian metric g on M. It is invariant under the geodesic flow. Analogously, on anti-de Sitter space we define $T^1\mathrm{AdS}_3 \subset T\mathrm{AdS}_3$; notice that each fiber $T^1_x\mathrm{AdS}_3$ is a non-compact surface. The space-like geodesics in $T^1\mathrm{AdS}_3$ are precisely those geodesics connecting two points in the conformal boundary $\partial\mathrm{AdS}_3 = \mathbb{S}^1 \times \mathbb{S}^1$.

The work [DGM25] provides a self-contained proof of the Axiom A property of the space-like geodesic flow φ_t on $T^1M = T^1(\Gamma\backslash\Omega_\Gamma)$ (which follows from [DMS25a]). One issue here is that the flow on T^1M is not complete: The flowout

$$\widetilde{\mathcal{M}} := \bigcup_{t\in\mathbb{R}} \varphi_t(T^1\Omega_\Gamma) \subset T^1\mathrm{AdS}_3$$

is strictly larger than $T^1\Omega_\Gamma$, and the first main observation in [DGM25] is that Γ acts properly discontinuously and freely on $\widetilde{\mathcal{M}}$. Thus,

$$\mathcal{M} := \Gamma\backslash\widetilde{\mathcal{M}}$$

is a smooth manifold to which the complete flow φ_t descends, extending the incomplete space-like geodesic flow on $T^1M \subset \mathcal{M}$. Now $(\mathcal{M}, \varphi_t)$ is an Axiom A system, which we call the *extended space-like geodesic flow*. The non-wandering set $\mathcal{K}$ of this flow, on which the essential dynamics takes place, is contained in T^1M.

4.2.1 Spectral theory of the generator of the extended space-like geodesic flow

First of all, one observes that for $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ we can define the operator

$$R_X(\lambda) : L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M}), \quad (R_X(\lambda)f)(x) := \int_0^\infty e^{-t\lambda} f(\varphi_{-t}(x))dt,$$

which is also continuous as an operator

$$R_X(\lambda) : C_c^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M}),$$

where $\mathcal{D}'(\mathcal{M})$ denotes the space of distributions on $\mathcal{M}$. This defines a holomorphic family of operators on $\{\Re\lambda > 0\} \subset \mathbb{C}$ which are resolvents of $-X$ in the sense that

$$(X+\lambda)(R_X(\lambda)f) = R_X(\lambda)((X+\lambda)f) = f \qquad \forall f \in C_c^\infty(\mathcal{M}),$$

$(X+\lambda)$ mapping $\mathcal{D}'(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ (resp. $C_c^\infty(\mathcal{M}) \to C_c^\infty(\mathcal{M})$) on the left (resp. on the right). The resolvent $R_X(\lambda)$ has the property that it propagates supports forward.

Theorem 11 ([DGM25, Theorem A]) *The resolvent* $R_X(\lambda) : C_c^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ *admits a meromorphic continuation from* $\mathrm{Re}(\lambda) > 0$ *to* $\lambda \in \mathbb{C}$ *with poles of finite rank, it has a simple pole at* $\lambda_0 := h_{\mathrm{top}} - 2$ *where* h_{top} *is the topological entropy of the flow* φ_t *on the non-wandering set* $\mathcal{K}$*, and no other pole in* $\mathrm{Re}(\lambda) > \lambda_0 - \varepsilon$ *for some* $\varepsilon > 0$*, and the residue at* λ_0 *is a rank* 1 *operator of the form*

$$\Pi_{\lambda_0} := \mathrm{Res}_{\lambda_0} R_X(\lambda) = u \otimes v$$

with u *a measure supported on* $\mathcal{K}_+$ *and* v *a measure supported on* $\mathcal{K}_-$*, where*

$$\mathcal{K}_\pm := \{x \in \mathcal{M} \mid \overline{\{\varphi_{\mp t}(x) \mid t \in [0,\infty)\}} \subset \mathcal{M} \text{ is compact}\}.$$

More generally, we consider a $\mathbb{C}$-vector bundle $\mathcal{E}$ over $\mathcal{M}$ and a lift of X to a first order differential operator $\mathbf{X}$ acting on smooth sections of e, which contains the case of the Lie derivative $\mathcal{L}_X$ of differential forms. The poles of the meromorphic resolvent are called *Ruelle resonances* of $\mathbf{X}$. They form a discrete spectrum in the complex plane intrinsic to $\mathbf{X}$, and each Ruelle resonance comes with finite-dimensional spaces of (generalized) *resonant states*.

4.2.2 Spectrum of the pseudo-Riemannian Laplacian

The pseudo-Riemannian Laplacian $\Box_g$ on the quasi-Fuchsian Lorentzian manifold $M = \Gamma\backslash\Omega_\Gamma$ is formally self-adjoint when acting on $C_c^\infty(M)$, but it is not clear if it is essentially self-adjoint on $L^2(M)$. On AdS_3 however, it is the case that $\Box_g$ has a self-adjoint extension with spectrum $\sigma(\Box_g) = [1,\infty) \cup \{1 - n^2 \mid n \in \mathbb{N}^*\}$ and we show in [DGM25, Sec. 3] that it has a resolvent

$$(\Box_g + \lambda(\lambda+2))^{-1} : L^2(\mathrm{AdS}_3) \to L^2(\mathrm{AdS}_3)$$

inverting $\Box_g + \lambda(\lambda+2)$ which is meromorphic in $\mathrm{Re}(\lambda) > -1$ with simple poles of infinite multiplicity at $\mathbb{N}$ (corresponding to discrete series of $\mathrm{SL}(2,\mathbb{R})$), and which extends meromorphically to $\mathbb{C}$ as a map $C_c^\infty(\mathrm{AdS}_3) \to \mathcal{D}'(\mathrm{AdS}_3)$ with simple poles at $\mathbb{Z}$ and infinite multiplicity. Its integral kernel has the form $F_\lambda(-q(x,x'))$ for some explicit function F_λ, see [DGM25, Prop. 3.1]. The integral kernel $F_\lambda(-q(x,x'))$ can be slightly modified to remove the poles at $\lambda \in \mathbb{Z}$ to another integral kernel $F_\lambda^h(-q(x,x'))$ solving

$$(\Box_g + \lambda(\lambda+2))F_\lambda^h(-q(x,x')) = \delta_{x=x'}$$

in the region $\{(x,x') \in \mathrm{AdS}_3^2 \mid -q(x,x') > -1\}$, with F_λ^h holomorphic in $\lambda \in \mathbb{C}$ and $F_\lambda^h(\zeta) = C(\lambda)F_\lambda(\zeta)$ in the region $\zeta > 1$ for some explicit holomorphic function $C(\lambda)$ (see [DGM25, Prop. 3.1] for the explicit formula for F_λ^h).

It turns out that for $x, x' \in \Omega_\Gamma$ the operator $R^h_{\Box_g}(\lambda)$ with integral kernel $F_\lambda^h(-q(x,x'))$ inverts $(\Box_g + \lambda(2+\lambda))$ on Ω_Γ. Motivated by the Riemannian case, we propose as a definition for the resolvent of $\Box_g$ on M the operator $R_{\Box_g} : C_c^\infty(M) \to \mathcal{D}'(M)$ with the following integral kernel: For $x, x' \in \Omega_\Gamma$ and $\mathrm{Re}(\lambda) > -1$

$$R_{\Box_g}(\lambda; x, x') := \sum_{\gamma \in \Gamma} F_\lambda^h(-q(x, \gamma x')) \tag{7}$$

which descends as an $L^1_{\mathrm{loc}}(M \times M)$ integral kernel that solves $(\Box_g + \lambda(2+\lambda))R_{\Box_g}(\lambda) f = f$ for all $f \in C_c^\infty(M)$.

The following is a shortened version of the main result of [DGM25] on the spectral theory of the pseudo-Riemannian Laplacian $\Box_g$ as well as an associated quantum-classical correspondence.

Theorem 12 ([DGM25, Theorem C]) *The operator $R_{\Box_g}(\lambda)$ defined by (7) admits a meromorphic extension to $\mathbb{C}$ as a continuous operator*

$$R_{\Box_g}(\lambda) : C_c^\infty(M) \to \mathcal{D}'(M)$$

satisfying $(\Box_g + \lambda(\lambda+2))R_{\Box_g}(\lambda) f = f$ for all $f \in C_c^\infty(M)$, with poles of finite rank, and for all $f_1, f_2 \in C_c^\infty(M)$ one has

$$\langle R_{\Box_g}(\lambda) f_1, f_2 \rangle = \frac{1}{2}\langle R_X(\lambda)\pi^* f_1, \pi^* f_2 \rangle - \frac{1}{2}\langle R_X(\lambda+2)\pi^* f_1, \pi^* f_2 \rangle + \langle H(\lambda) f_1, f_2 \rangle,$$

where $R_X(\lambda)$ is the meromorphic resolvent from Theorem 11 and $H(\lambda)$ is a holomorphic family of continous operators $C_c^\infty(M) \to \mathcal{D}'(M)$.

This result also implies that the quantum resonances are contained in translates of the Ruelle resonance for the spacelike geodesic flow, and the quantum resonant states are necessarily pushforwards of Ruelle resonant states, in a way comparable to the Riemannian case [DFG15, GHW18].

Finally, in [DGM25, Sec. 7.1], we describe completely the quantum resonances and resonant states when the group Γ is Fuchsian, i.e., the Lorentzian 3-manifold becomes

$$M = (-\pi/2, \pi/2)_\theta \times \Sigma, \quad g = -d\theta^2 + \cos(\theta)^2 g_\Sigma,$$

with $(\Sigma, g_\Sigma) = \Gamma\backslash\mathbb{H}^2$ a closed oriented surface and $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ co-compact.

5 Outlook and open questions

The link between microlocal analysis and higher Teichmüller theory established by the works [DMS25a, DMS25b] is still largely unexplored. For example, it is so far unclear in which generality higher rank (partially) hyperbolic (multi-)flows can be defined that possess a good notion of dynamical resonance spectrum, and how this allows to characterise interesting classes of discrete subgroups of Lie groups larger than the class of Anosov subgroups. This question will be addressed in forthcoming volumes of the series of papers starting with [DMS25a, DMS25b]. For various related questions and a comprehensive study of multiflows associated to Anosov subgroups via the domains of discontinuity introduced in [DMS25a, DMS25b], see also the recent work [GBLW26].

From a mathematical physics perspective, the quantum-classical correspondence established in [DGM25] is expected to exist in much more general situations – essentially whenever a "quantum analogue" of the classical system represented by the dynamically complete geometric structure associated to an Anosov representation can be identified. Moreover, even in the three-dimensional Lorentzian quasi-Fuchsian case studied in [DGM25], the spectral properties of the quantum operator given by the pseudo-Riemannian Laplacian $\Box$ are still mysterious in many aspects and deserve further study.

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References

Ben04. Y. Benoist, *Convexes divisibles. I*, Algebraic groups and arithmetic, Tata Inst. Fund. Res., Mumbai, 2004, pp. 339–374 (French).

Ben08. ______, *A survey on divisible convex sets*, Geometry, analysis and topology of discrete groups, Adv. Lect. Math. (ALM), vol. 6, Int. Press, Somerville, MA, 2008, pp. 1–18.

BL07. O. Butterley and C. Liverani, *Smooth Anosov flows: correlation spectra and stability*, J. Mod. Dyn **1** (2007), no. 2, 301–322.

Bor16. D. Borthwick, *Spectral theory of infinite-area hyperbolic surfaces*, 2nd edition ed., Prog. Math., vol. 318, Basel: Birkhäuser/Springer, 2016.

BPS19. J. Bochi, R. Potrie, and A. Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 11, 3343–3414.

BWS21. Y. Borns-Weil and S. Shen, *Dynamical zeta functions in the nonorientable case*, Nonlinearity **34** (2021), no. 10, 7322–7334.

CS24. L. Carvajales and F. Stecker, *Anosov representations acting on homogeneous spaces: Domains of discontinuity*, Advances in Mathematics **459** (2024), 110022.

DFG15. S. Dyatlov, F. Faure, and C. Guillarmou, *Power spectrum of the geodesic flow on hyperbolic manifolds*, Analysis & PDE **8** (2015), no. 4, 923–1000.

DG16. S. Dyatlov and C. Guillarmou, *Pollicott–Ruelle resonances for open systems*, Ann. Henri Poincaré **17** (2016), no. 11, 3089–3146.

DG18. ______, *Afterword: Dynamical zeta functions for Axiom A flows*, Bull. Amer. Math. Soc. (2018), no. 55, 337–342.

DGK18. J. Danciger, F. Guéritaud, and F. Kassel, *Convex cocompactness in pseudo-Riemannian hyperbolic spaces.*, Geom. Dedicata **192** (2018), 87–126.

DGM25. B. Delarue, C. Guillarmou, and D. Monclair, *Spectra of Lorentzian quasi-Fuchsian manifolds*, 2025, preprint arXiv:2504.21762, to appear in Comm. Math. Phys.

DMS25a. B. Delarue, D. Monclair, and A. Sanders, *Locally Homogeneous Axiom A Flows I: Projective Anosov Subgroups and Exponential Mixing*, Geom. Funct. Anal. **35** (2025), 673–735.

DMS25b. ______, *Locally homogeneous Axiom A flows II: geometric structures for Anosov subgroups*, 2025, preprint arXiv:2502.20195.

DZ16. S. Dyatlov and M. Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, Ann. Sci. Ec. Norm. Supér **49** (2016), 543–577.

FS11. F. Faure and J. Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Commun. Math. Phys. **308** (2011), no. 2, 325–364, DOI.

GBLW26. Y. Guedes Bonthonneau, T. Lefeuvre, and T. Weich, *The spectrum of Anosov representations*, 2026, preprint arXiv:2603.24519.

GGKW17. F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, *Anosov representations and proper actions*, Geom. Topol. **21** (2017), no. 1, 485–584.

GHW18. C. Guillarmou, J. Hilgert, and T. Weich, *Classical and quantum resonances for hyperbolic surfaces*, Mathematische Annalen **370** (2018), no. 3, 1231–1275.

GW08. O. Guichard and A. Wienhard, *Convex foliated projective structures and the Hitchin component for* $\mathrm{PSL}_4(\mathbf{R})$, Duke Math. J. **144** (2008), no. 3, 381–445.

GW12. ______, *Anosov representations: domains of discontinuity and applications.*, Invent. Math. **190** (2012), no. 2, 357–438.

GZ03. C.R. Graham and M. Zworski, *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), 89–118, DOI.

Had18. C. Hadfield, *Ruelle and quantum resonances for open hyperbolic manifolds*, Int. Mathem. Res. Not. **2020** (2018), no. 5, 1445–1480.

JSB00. M. Joshi and A. Sa Barreto, *Inverse scattering on asymptotically hyperbolic manifolds*, Acta Math. **184** (2000), 41–86.

Kas18. F. Kassel, *Geometric structures and representations of discrete groups*, Proceedings of the international congress of mathematicians 2018, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume II. Invited lectures, Hackensack, NJ: World Scientific; Rio de Janeiro: Sociedade Brasileira de Matemática (SBM), 2018, pp. 1115–1151.

KLP17. M. Kapovich, B. Leeb, and J. Porti, *Anosov subgroups: dynamical and geometric characterizations*, Eur. J. Math. **3** (2017), no. 4, 808–898.

KLP18. ———, *Dynamics on flag manifolds: domains of proper discontinuity and cocompactness*, Geom. Topol. **22** (2018), no. 1, 157–234.

KP22. F. Kassel and R. Potrie, *Eigenvalue gaps for hyperbolic groups and semigroups*, J. Mod. Dyn. **18** (2022), 161–208.

Lab06. F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

Led95. F. Ledrappier, *Structure au bord des variétés à courbure négative*, Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, Sémin. Théor. Spectr. Géom., vol. 13, Univ. Grenoble I, Saint-Martin-d'Hères, 1995, pp. 97–122 (French).

Liv04. C. Liverani, *On contact Anosov flows*, Ann. Math. (2) **159** (2004), no. 3, 1275–1312.

Mes07. G. Mess, *Lorentz spacetimes of constant curvature*, Geom. Dedicata **126** (2007), 3–45.

MM87. R. R. Mazzeo and R. B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260–310.

NR24. A. Nolte and J. M. Riestenberg, *Concave Foliated Flag Structures and the $SL_3(\mathbb{R})$ Hitchin Component*, 2024, preprint arXiv:2407.06361.

PP01. S.J. Patterson and P.A. Perry, *The divisor of Selberg's zeta function for Kleinian groups*, Duke Math. J. **106** (2001), no. 2, 321–390.

Qui02a. J.-F. Quint, *Cônes limites des sous-groupes discrets des groupes réductifs sur un corps local*, Transform. Groups **7** (2002), no. 3, 247–266 (French).

Qui02b. ———, *Divergence exponentielle des sous-groupes discrets en rang supérieur*, Comment. Math. Helv. **77** (2002), no. 3, 563–608 (French).

Sam14. A. Sambarino, *Quantitative properties of convex representations*, Comment. Math. Helv. **89** (2014), no. 2, 443–488.

Sam24. ———, *A report on an ergodic dichotomy*, Ergodic Theory Dynam. Systems **44** (2024), no. 1, 236–289.

Sma67. S. Smale, *Differentiable dynamical systems. With an appendix to the first part of the paper: "Anosov diffeomorphisms" by John Mather*, Bull. Am. Math. Soc. **73** (1967), 747–817.

Wie18. A. Wienhard, *An invitation to higher Teichmüller theory*, Proceedings of the international congress of mathematicians 2018, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume II. Invited lectures, Hackensack, NJ: World Scientific; Rio de Janeiro: Sociedade Brasileira de Matemática (SBM), 2018, pp. 1013–1039.